Lecture 23 (3/4/22)

- Finish pf of Runge's Thm from Lecture 22 notes:

- Express $f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(\zeta)\,d\zeta}{\zeta - z}$, $\gamma_j \subset\subset G \setminus K$. ✓

- Approx. by rational functions using Riemann sum

$$R(z) = \sum_{k=1}^{m} b_k \frac{1}{z - a_k}, \quad a_k \in \gamma_j \ ✓$$

- Approximate $\frac{1}{z-a}$, $a \in G \setminus K$ by rational function w/ poles in E. (Prop 3 from LN 22).

Uses property of exhaustion K_n previously constructed.

Important Corollaries.

1. Let $G \subseteq \mathbb{C}$ be a region and $E \subseteq \mathbb{C}_\infty \setminus G$ that meets every component. If $R(E)$ denotes rational functions w/ poles in E, then $R(E)$ is dense in $H(G)$.
2. If $\mathbb{C}_\infty \setminus G$ is connected, then polynomials are dense in $H(G)$.

<u>Def ①</u>. If $K \subset\subset G$, then the <u>polynomial hull</u> $\hat{K}$ of K is

$$\hat{K} = \left\{ z \in \mathbb{C} : |p(z)| \le \sup_K |p|, \ \forall \text{polynomials } p(z) \right\}$$

Note that if $p_n \to f$ unif. on K, for some f, then $p_n \to \hat{f}$ unif. on $\hat{K}$, where $\hat{f} = f$ on $K \subseteq \hat{K}$.

<u>Prop 1.</u> $\hat{K}$ is compact and $\mathbb{C}_\infty \setminus \hat{K}$ is connected.

<u>Pf.</u> $\hat{K}$ is clearly closed and bounded, so compact. If $\mathbb{C}_\infty \setminus \hat{K}$ were <u>not</u> connected, then $\hat{\mathbb{C}}_\infty \setminus \hat{K}$ would have a bounded component H_0.

Then $\partial H_0 \subseteq \hat{K}$. Thus, if p is a polynomial and $z \in H_0$, then by Max Mod Thm

$$p(z) \le \sup_{\partial H_0} |p| \le \sup_{\hat{K}} |p| \underset{\text{def.}}{\le} \sup_{K} |p|$$

$\Rightarrow z \in \hat{K}$, a contradiction. Thus, $\mathbb{C}_\infty \setminus \hat{K}$ can only have the unbounded component. ▨

Equivalent conditions for simple connectedness.

We mention here only some of them. The total list is Thm 2.2 in Ch. VIII. The pf is given in Conway.

<u>Thm</u>. TFAE for a region $G \subseteq \mathbb{C}$.

(i) G is simply connected.

(ii) $\mathbb{C}_\infty \setminus G$ is connected.

(iii) polynomials are dense in $H(G)$.

(iv) Every $f \in H(G)$ has a primitive.

(v) If $f \neq 0$ in G, then there is a branch of $\log f$.

(vi) G is homeomorphic to $\mathbb{D} = \{|z|<1\}$.